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LETTER TO THE EDITOR

On the algebraic Bethe ansatz for the XXX spin chain: creation operators 'beyond the equator'

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Abstract
Considering the XXX spin-1/2 chain in the framework of the algebraic Bethe ansatz, we make the following short comment: the product of the creation operators corresponding to the recently found solution of the Bethe equations 'on the wrong side of the equator' [1] is just zero (not only its action on the pseudovacuum).

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Consider the periodic XXX spin-1/2 chain with N sites in the framework of the algebraic Bethe ansatz (ABA) (see, for example, [2]). Let us introduce the Lax operator acting in the two-dimensional local quantum space $h_n = \mathbb{C}^2$ and in the two-dimensional auxiliary space $V = \mathbb{C}^2$:

$$L_n(x) = \begin{pmatrix} x + \mathrm{i}s_n^3 & \mathrm{i}s_n^- \\ \mathrm{i}s_n^+ & x - \mathrm{i}s_n^3 \end{pmatrix} \tag{1}$$

where s^i are operators of spin 1/2 and x is an arbitrary complex number (the spectral parameter). The monodromy matrix is the ordered product over all sites:

$$T(x) = L_N(x)L_{N-1}(x)\ldots L_1(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \tag{2}$$

where $A(x), B(x), C(x), D(x)$ are operators acting in the full quantum space $H = \otimes\prod_{n=1}^{N} h_n$. In the framework of ABA, one looks for the eigenvectors of the transfer matrix

$$\hat{t}(x) = \mathrm{tr}\, T(x) = A(x) + D(x)$$

in the form

$$\Phi(\{x\}) = B(x_1)B(x_2)\ldots B(x_l)\Omega \tag{3}$$

where $\Omega = \prod_{n=1}^{N} \omega_n$, $s_n^+ \omega_n = 0$. It follows from the intertwining relations for the monodromy matrices that vector (3) will be an eigenvector of the transfer matrix when the parameters $x_1, \ldots, x_l$ satisfy the Bethe equations:

$$\left(\frac{x_j + \mathrm{i}/2}{x_j - \mathrm{i}/2}\right)^N = \prod_{k \neq j}^{l} \frac{x_j - x_k + \mathrm{i}}{x_j - x_k - \mathrm{i}} \qquad (j = 1, 2, \ldots, l). \tag{4}$$

Let us denote vectors of the form (3) with parameters x_j $(j = 1, 2, \ldots, l)$ satisfying system (4) by $\Phi(\{x\}_B)$. It is well known that vectors $\Phi(\{x\}_B)$ are the highest weights vectors with respect to $SU(2)$ generated by J^i, i.e.

$$J^+ \Phi(\{x\}_B) = 0 \tag{5}$$

and

$$J^3 \Phi(\{x\}_B) = (N/2 - l) \Phi(\{x\}_B) \tag{6}$$

where J^3, $J^\pm$ are operators of the total spin. It is clear that if $l > N/2$ then $\Phi(\{x\}_B) = 0$. The solution $\{x\}$ of (4) with $l \leqslant N/2$ defines the polynomial $q(x)$ of the degree l, whose roots are $\{x\}$

$$q(x) = \prod_{j=1}^{l} (x - x_j). \tag{7}$$

Let $t(x)$ be the eigenvalue of transfer matrix $\hat{t}(x)$ corresponding to the eigenvector $\Phi(\{x\}_B)$, i.e. $\hat{t}(x)\Phi(\{x\}_B) = t(x)\Phi(\{x\}_B)$. It is a polynomial of degree N. Then the polynomials $t(x)$ and $q(x)$ satisfy the Baxter equation [4] (we consider the case of the simple roots):

$$t(x)q(x) = (x - \mathrm{i}/2)^N q(x + \mathrm{i}) + (x + \mathrm{i}/2)^N q(x - \mathrm{i}). \tag{8}$$

In [1] it was shown that there exists a polynomial $p(x)$ of degree $N - l + 1$ with roots satisfying the Bethe equation (4)[4] and

$$t(x)p(x) = (x - \mathrm{i}/2)^N p(x + \mathrm{i}) + (x + \mathrm{i}/2)^N p(x - \mathrm{i}) \tag{9}$$

with the same $t(x)$ as in (8). Actually, there exists a one-parametric family of such polynomials

$$p(x, \alpha) = p(x) + \alpha q(x) \tag{10}$$

so there is the one-parametric family of sets of parameters $(\{x\})$—the zeros of $p(x, \alpha)$ which belongs to the 'beyond the equator' case. Let us denote these zeros as $x_i(\alpha)$ (it is clear that the zeros of polynomial (10) depend on α). Now consider the creation operator

$$\mathbf{B}(\alpha) = B(x_1(\alpha))B(x_2(\alpha)) \ldots B(x_{N-l+1}(\alpha))$$

corresponding to the beyond the equator case. The following statement is valid:

Theorem.

$$\mathbf{B}(\alpha) = 0. \tag{11}$$

Proof. Consider its action on the basis constructed using the Bethe vectors (in the case of finite $\{x\}$, Bethe vectors are the highest weights); to obtain the rest of the eigenvectors, we use the operator J^- (it commutes with the transfer matrix), which can also be considered as a creation operator, since $B(x) = x^{N-1}(J^- + o(1/x))$ when $x \to \infty$; on the hypothesis of the

[4] See also the interesting discussion of the 'beyond equator' solution in [5].

completeness of the Bethe ansatz (see [2, 3]). We immediately see that its action is zero due to $[B(x), B(y)] = 0$, $[B(x), J^-] = 0$ and $\mathbf{B}(\alpha)\Omega = 0$. So, the action of $\mathbf{B}(\alpha)$ is zero onto each vector of the basis, then

$$\mathbf{B}(\alpha) = 0. \tag{12}$$

□

To see that this fact is nontrivial let us consider the concrete examples. Let us first analyse the structures of the product of B-operators. We have the following commutation relation:

$$[B(x), J^3] = B(x) \tag{13}$$

so for the product of l B-operators

$$\mathrm{e}^{\beta J^3} B(x_1)\dots B(x_l)\,\mathrm{e}^{-\beta J^3} = \mathrm{e}^{-l\beta} B(x_1)\dots B(x_l) \tag{14}$$

and we see that each term of the expansion of this product necessarily contains the products of l-operators s_j^- with different j, since $(s^-)^2 = 0$ (so if $l > N$, this product is zero). At $N = 6$, product of four B-operators ($l = 4 > 3$, i.e. this is the 'beyond the equator' case): $B(x_1)B(x_2)B(x_3)B(x_4)$, as was shown by the explicit construction of this product, contains terms proportional to $s_2^+s_1^-s_3^-s_4^-s_5^-s_6^-$, $s_3^+s_1^-s_2^-s_4^-s_5^-s_6^-$, $s_4^+s_1^-s_2^-s_3^-s_5^-s_6^-$ and $s_5^+s_1^-s_2^-s_3^-s_4^-s_6^-$ with non-zero coefficients polynomials in x_1, x_2, x_3, x_4. The appearance of such terms is not excluded by (14). Their action on the vacuum Ω is identically zero, so if x_1, x_2, x_3, x_4 satisfy the system (4), then the action of these terms become zero; the only terms whose action on the vacuum Ω is nonzero are the terms proportional to $s_1^-s_2^-s_3^-s_4^-$. Such solutions do exist, for example, roots of the polynomial

$$p(x) = x^4 - \frac{6}{\sqrt{13}}x^3 + x^2 - \frac{9}{16} \tag{15}$$

satisfy system (4). This solution corresponds to the total spin $J = 0$, the eigenvalue of the transfer matrix

$$t(x) = 2x^6 + \frac{9}{2}x^4 + \frac{23}{8}x^2 - \frac{3}{\sqrt{13}}x - \frac{1}{32}.$$

The corresponding eigenvector can be constructed, using the roots of the polynomial

$$q(x) = x^3 + \frac{1}{12}x + \frac{1}{4\sqrt{13}} \tag{16}$$

and the one-parametric family $\mathbf{B}(\alpha) = 0$ corresponds to the roots of the polynomial

$$p(x,\alpha) = x^4 - \frac{6}{\sqrt{13}}x^3 + x^2 - \frac{9}{16} + \alpha\left(x^3 + \frac{1}{12}x + \frac{1}{4\sqrt{13}}\right). \tag{17}$$

The products of l-operators $B(x)$ with $l > N/2$, considered above, are not used for the construction of the eigenvectors of the transfer matrix. However, we would like to emphasize that there is another important example, when the product of B-operators corresponding to the case $l \leqslant N/2$ is zero:

$$B(-\mathrm{i}/2)B(\mathrm{i}/2) = 0.$$

This product corresponds to the polynomial

$$q(x) = x^2 + 1/4$$

and

$$t(x) = (x + \mathrm{i}/2)^{N-1}(x - 3/2\mathrm{i}) + (x - \mathrm{i}/2)^{N-1}(x + 3/2\mathrm{i})$$

and at even $N \geqslant 4$, corresponds to some eigenvector of the transfer matrix. In paper [6], it was shown how one can use the ABA to construct the eigenvector corresponding to this exceptional solution. If we consider the following vector, $B(-\mathrm{i}/2 + \epsilon + 2\mathrm{i}\epsilon^N)B(\mathrm{i}/2 + \epsilon)\Omega$ at $\epsilon \to 0$, then

$$B(-\mathrm{i}/2 + \epsilon + 2\,\mathrm{i}\epsilon^N)B(\mathrm{i}/2 + \epsilon)\Omega = \epsilon^N \Phi(\{-\mathrm{i}/2,\ \mathrm{i}/2\}) + O(\epsilon^{N+1})$$

where $\Phi(\{-\mathrm{i}/2,\ \mathrm{i}/2\})$ is the desired eigenvector. The proof of our statement holds true in this case too, since we use again only B-operators to construct this eigenvector.

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